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The behaviour of a laminar compressible boundary layer on a cold wall near a point of zero skin friction

By J. BUCKMASTER

Mathematics Department, New York University, University Heights, N.Y.

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It is shown that the expansion assumed by Stewartson to describe the flow close to separation in a compressible boundary layer is incomplete. When the wall is cold an infinity of new terms involving $\log \xi$, $\log\log \xi$ and their products and quotients must be added at each algebraic stage. The skin friction then vanishes like $x^{\frac{1}{2}} \ln x$ where x is the distance to separation. None of the coefficients of the logarithmic terms are arbitrary and in particular the first two terms in the expansion of the skin friction are known if the heat transfer is given at separation. Convergence is so slow, however, that this is of no practical value.

1. Introduction

The behaviour of an incompressible boundary layer at a point of zero skin friction has been firmly established by the work of Goldstein (1948), Stewartson (1958) and Terrill (1960). The skin friction vanishes like $x^{\frac{1}{2}}$ where x is the distance to separation, and the structure close to the wall is described by a series in powers of $x^{\frac{1}{4}}$ with coefficients that are functions of $\eta \equiv y/(2x)^{\frac{1}{4}}$. At various stages, terms in $x^{\frac{1}{4}n} \log x$ also have to be included, this being the fundamental contribution of Stewartson. There are two basic assumptions in the development of what will be called the Goldstein–Stewartson expansion, namely that η is the appropriate similarity variable and that the various complicated functions of η that are generated all behave algebraically when η is very large.

In the case of compressible flow the matter is less satisfactory, since Stewartson (1962), using the same approach as his 1958 paper, was unable to find anything but a regular expansion at separation when the heat-transfer is non-zero. This result was not contradicted by the best numerical work of the time, but recent numerical work of Merkin (1969) for a cold wall shows singular behaviour difficult to distinguish numerically from the square root. P. G. Williams of University College, London, informs me that he also has found singular behaviour for both hot and cold walls.

In this paper Stewartson's approach is re-examined in an attempt to resolve this contradiction between the analysis and the numerical work. Apparently a self-consistent expansion can be found, valid for a cold wall, if additional terms involving log log are permitted.

The equations to be studied are (Stewartson 1962)

$$\frac{\partial^3 f}{\partial \eta^3} - 3f\frac{\partial^2 f}{\partial \eta^2} + 2\left(\frac{\partial f}{\partial \eta}\right)^2 + \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi}\right) = (1+g)\sum_{r=0}^{\infty} P_r \xi^{4r}, \tag{1.1}$$

$$\frac{\partial^2 g}{\partial \eta^2} - 3f\frac{\partial g}{\partial \eta} + \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta}\right) = 0, \tag{1.2}$$

with boundary conditions

$$\left.\begin{aligned} f(\xi, 0) = 0, \quad \partial f(\xi, 0)/\partial \eta = 0, \\ g(\xi, 0) = \sum_{r=1}^{\infty} \xi^{4r} S_r. \end{aligned}\right\} \tag{1.3}$$

In addition it is required that neither of the dependent variables diverge exponentially for large η.

Here,
$$\xi = (-X/l)^{\frac{1}{4}}, \quad \eta = (Re)^{\frac{1}{2}}(l/-4X)^{\frac{1}{4}}\, Y,$$
$$\psi = 2^{\frac{3}{2}}(-X/l)^{\frac{3}{4}} f(\xi, \eta), \quad S = t_0 + (1+t_0)\, g(\xi, \eta),$$

where X is the distance to separation, Y measures distance from the wall, ψ is the stream function and S is related to the absolute temperature. l is a characteristic length obtained from the pressure gradient and t_0 is the value of S at the separation point.

A solution to equations (1.1), (1.2) is sought in the form

$$f(\xi, \eta) = \sum_{n=0}^{\infty} f_n(\eta, \xi)\, \xi^n, \tag{1.4}$$

$$g(\xi, \eta) = \sum_{n=0}^{\infty} g_n(\eta, \xi)\, \xi^n, \tag{1.5}$$

where the ξ dependence of the f_n, g_n is logarithmic. More precisely

$$\lim_{\xi \to 0} \xi^{1+\alpha}\frac{\partial f_n}{\partial \xi} = \begin{cases} 0 & \text{when} \quad \alpha > 0, \\ \infty & \text{when} \quad \alpha < 0, \end{cases}$$

and similarly for the g_n. Consider now the derivative of (1.4),

$$\xi\frac{\partial f}{\partial \xi} = \sum_{n=0}^{\infty}\left(nf_n + \xi\frac{\partial f_n}{\partial \xi}\right)\xi^n,$$

which we write as
$$\xi\frac{\partial f}{\partial \xi} = \sum_{n=0}^{\infty} F_n\, \xi^n \tag{1.6}$$

and similarly
$$\xi\frac{\partial g}{\partial \xi} = \sum_{n=0}^{\infty} G_n\, \xi^n. \tag{1.7}$$

Equations (1.4)–(1.7) can be substituted into (1.1), (1.2) and the coefficients of powers of ξ equated as if the f_n, g_n, F_n, G_n were ξ independent. With the algebraic balance established and an infinite number of equations generated, one for each of the f_n, g_n we can then consider the 'logarithmic' expansion of each of these equations. The algebraic balance when $n \neq 0$ leads to

$$g_n'' - 3f_0 g_n' - F_0 g_n' + f_0' G_n = \sum_{m=1}^{n}\left[3f_m g_{n-m}' - f_m' G_{n-m} + F_m g_{n-m}'\right], \tag{1.8}$$

$$f_n''''-3f_0f_n''+f_0'F_n''-3f_0''f_n-f_0''F_n+4f_0'f_n'+f_n'F_0'-f_n''F_0$$
$$=\sum_{m=1}^{n-1}[3f_mf_{n-m}''-f_m'F_{n-m}'+f_m''F_{n-m}-2f_m'f_{n-m}']+P_{\frac{1}{4}n}+\sum_{m=0}^{\infty}g_mP_{\frac{1}{4}(n-m)}, \quad (1.9)$$

where the P_i are non-zero only for i a non-negative integer. The primes in equations (1.8), (1.9) denote derivatives with respect to η.

If f_0 and g_0 are assumed to be ξ independent they satisfy

$$g_0''-3f_0g_0'=0, \quad (1.10)$$

$$f_0''''-3f_0f_0''+2f_0'^2=1+g_0 \quad (1.11)$$

and Stewartson (1962) has given arguments from which it may be concluded that the appropriate solutions are

$$g_0=0, \quad f_0=\tfrac{1}{6}\eta^3. \quad (1.12)$$

The leading term in the temperature expansion is a constant and the velocity is parallel to the wall with a parabolic distribution.

2. Stewartson's analysis

Stewartson (1962) proceeded with the expansion by supposing that f_1, g_1 are ξ independent. Then,

$$g_1''-\tfrac{1}{2}\eta^3g_1'+\tfrac{1}{2}\eta^2g_1=0, \quad (2.1)$$

$$f_1''''-\tfrac{1}{2}\eta^3f_1''+\tfrac{5}{2}\eta^2f_1'-4\eta f_1=g_1P_0=g_1, \quad (2.2)$$

with solutions

$$g_1=B_1\eta, \quad f_1=\alpha_1\eta^2+\tfrac{1}{24}B_1\eta^4. \quad (2.3)$$

Continuing in like manner

$$g_2''-\tfrac{1}{2}\eta^3g_2'+\eta^2g_2=2\alpha_1B_1\eta^2. \quad (2.4)$$

$$f_2''''-\tfrac{1}{2}\eta^3f_2''+3\eta^2f_2'-5\eta f_2=-4\alpha_1^2\eta^2+\tfrac{1}{3}\alpha_1B_1\eta^4+g_2. \quad (2.5)$$

Equation (2.4) has solution

$$g_2=2\alpha_1B_1(1-\bar{g}_2), \quad (2.6)$$

where $\bar{g}_2$ is the complementary function that equals one at the wall and is algebraic at infinity. There is such a function. The equation for f_2, on the other hand, presents a difficulty since an acceptable solution can only be found if

$$\int^{\infty} d\eta\, e^{-\frac{1}{8}\eta^4}(\eta^2-\tfrac{1}{10}\eta^6)[g_2+\tfrac{1}{3}\alpha_1B_1\eta^4-4\alpha_1^2\eta^2]=0. \quad (2.7)$$

The η^2 term does not contribute to this integral since η^5 is an appropriate particular integral for this term. Thus (2.7) is only satisfied if

$$\alpha_1B_1=0. \quad (2.8)$$

Numerical evidence (e.g. Merkin 1969) suggests that the heat transfer does not vanish (i.e. $B_1 \neq 0$) and although the expansion can be continued with the choice $\alpha_1=0$, the solution is then regular at separation, which possibility we reject (at least for a cold wall). Stewartson attempted to avoid the conclusion (2.8) by a device that in his 1958 paper successfully resolved a similar difficulty that Goldstein (1948) had encountered. Thus he wrote

$$f_1 = f_{10}\ln\xi + f_{11}, \quad g_1 = g_1. \tag{2.9}$$

g_1 cannot be modified since this would ultimately lead to an inhomogeneous equation of the type (2.1); this equation would only have a solution if the inhomogenity satisfied an integral condition, and we would then conclude that in fact the inhomogenity vanished. Stewartson quite rightly concluded that (2.9) would not work but he omits the details. They are instructive however.

g_1 is still given by (2.3); f_{10} satisfies the homogeneous equation for f_1 so that

$$f_{10} = \alpha_{10}\eta^2 \tag{2.10}$$

and f_{11} satisfies (2.2) (although $\xi\,\partial f_1/\partial\xi = f_{10}$, the simple solution (2.10) does not contribute to f_{11}) whence

$$f_{11} = \alpha_{11}\eta^2 + \tfrac{1}{24}B_1\eta^4. \tag{2.11}$$

The logarithmic term in f_1 induces logarithmic terms in f_2, g_2. Thus

$$g_2 = g_{20}\ln\xi + g_{21},$$

where

$$g_{20} = 2\alpha_{10}B_1(1-\bar{g}_2)$$

and

$$g''_{21} - \tfrac{1}{2}\eta^3 g'_{21} + \eta^2 g_{21} = 2\alpha_{11}B_1\eta^2 + B_1\alpha_{10}\eta^2\bar{g}_2.$$

The solution for g_{21} is

$$g_{21} = 2B_1\alpha_{11}(1-\bar{g}_2) - \frac{B_1\alpha_{10}}{2\cdot 8^{\frac{1}{4}}\Gamma(\frac{1}{4})}h_2(\eta), \tag{2.12}$$

where

$$h_2(\eta) \equiv \eta\int_0^\infty dq\left[(1+q)^{\frac{1}{2}}\frac{\ln(1+q)}{q^{\frac{5}{4}}}e^{-\eta^4/8q} - \frac{\ln q}{q^{\frac{3}{4}}}\right].$$

Three terms are needed to describe f_2

$$f_2 = f_{20}\ln^2\xi + f_{21}\ln\xi + f_{22},$$

where

$$f_{20} = \alpha_{20}\eta^2 - \tfrac{1}{15}\alpha_{10}^2\eta^5, \tag{2.13}$$

$$f'''_{21} - \tfrac{1}{2}\eta^3 f''_{21} + 3\eta^2 f'_{21} - 5\eta f_{21} = -8\alpha_{10}\alpha_{11}\eta^2 - 2\alpha_{10}^2\eta^2 + \tfrac{1}{3}B_1\alpha_{10}\eta^4 + \tfrac{1}{5}\alpha_{10}^2\eta^6 + 2\alpha_{10}B_1(1-\bar{g}_2). \tag{2.14}$$

$$f'''_{22} - \tfrac{1}{2}\eta^3 f''_{22} + 3\eta^2 f'_{22} - 5\eta f_{22} = -4\alpha_{11}^2\eta^2 - 2\alpha_{10}\alpha_{11}\eta^2 + g_{21} + \tfrac{1}{6}B_1\alpha_{10}\eta^4 + \tfrac{1}{3}B_1\alpha_{11}\eta^4 - \tfrac{1}{2}\eta^2 f'_{21} + \eta f_{21}. \tag{2.15}$$

Equation (2.14) only has a solution if an integral restraint similar to (2.7) is satisfied. Since

$$\bar{g}_2 = \frac{1}{8^{\frac{1}{4}}\Gamma(\frac{1}{4})}\eta\int_0^\infty dq\left[e^{-\eta^4/8q}\frac{(1+q)^{\frac{1}{2}}}{q^{\frac{5}{4}}} - \frac{1}{q^{\frac{3}{4}}}\right],$$

we find

$$\underline{\alpha_{10} = -B_1\frac{2\pi^{\frac{1}{2}}\Gamma(\frac{3}{4})}{[\Gamma(\frac{1}{4})]^3}}. \tag{2.16}$$

This result is underlined since it is not discarded in the sequel. It relates the leading term in the skin friction to the heat transfer at separation. Now α_{10} must be negative since the skin friction is positive just prior to separation.

Equation (2.16) can only be correct then if $B_1 > 0$ so that the temperature is increasing away from the wall. Henceforth the analysis will be restricted to this cold-wall case.

It might be thought that the integral restraint implied by (2.15) would establish a relation between α_{10} and α_{11}. However, because

$$f_{21} = -\tfrac{2}{15}\alpha_{10}\alpha_{11}\eta^5 + \text{terms independent of } \alpha_{11}$$

we have

$$\int_0^\infty d\eta\, e^{-\frac{1}{8}\eta^4}(\eta^2 - \tfrac{1}{10}\eta^6)\,[2B_1\alpha_{11}(1-\bar{g}_2) + \tfrac{1}{3}B_1\alpha_{11}\eta^4 + \tfrac{1}{5}\alpha_{10}\alpha_{11}\eta^6]$$
$$= \text{terms independent of } \alpha_{11} \quad (2.17)$$

and the left side of this equation vanishes because of the choice of α_{10}. Equation (2.17) then establishes another relation between α_{10} and B_1 that is not consistent with (2.16). Because of this, Stewartson concluded that (2.8) is correct.

3. The modified expansion

In this section it is shown how the difficulty of § 2 can be avoided by permitting additional terms in the expansion. Before doing this, however, it is worth mentioning that the author's original approach to this problem was not to seek a Goldstein–Stewartson expansion, but rather to treat the problem as a parameter perturbation following Kaplun's (1967) analysis of the incompressible case. In this approach perturbations to (1.12) are sought without any assumptions about the structure. This leads to partial differential equations and an eigenfunction which satisfies a certain non-linear integral equation with an Abel kernel. In order to determine the behaviour of the skin-friction at separation it is then necessary to find a local expansion for the eigenfunction. One such expansion was found, valid for a cold wall, and it is that expansion which we describe here, although in a different form. Although the needed terms were discovered in this fashion, an argument can be given within the present framework, as follows.

If we take the point of view that f_{10} is correct since the contradiction arose at the $O(1)$ stage in f_2, then we must seek an additional term, somewhat larger than f_{11}. This term must provide an additional inhomogeneity in the equation for f_{22}. Now in addition to other terms the equation for f_2 contains

$$-f_1'\xi\frac{\partial f_1'}{\partial \xi} + f_1''\xi\frac{\partial f_1}{\partial \xi}, \quad (3.1)$$

so that if we choose a term that satisfies

$$\ln\xi\,.\,\xi\,\partial f_1/\partial\xi \sim 1,$$

i.e. an $O(\ln\ln\xi)$ term, then the equation for f_{22} is changed. However (3.1) provides an $O(\eta^2)$ term which is not good enough but fortunately an $O(\ln\xi\ln\ln\xi)$ term is added to f_2 and it is the ξ derivative of this that resolves the difficulty.

Equation (2.9) is now replaced by

$$f_1 = f_{10}\ln\xi + f_{12}\ln\ln\xi + f_{11}.\dagger \tag{3.2}$$

f_{10} and f_{11} are still given by (2.10), (2.11), and

$$f_{12} = \alpha_{12}\eta^2. \tag{3.3}$$

Equation (3.2) is then an exact solution of the equation for f_1.

The $\ln\ln\xi$ term complicates the expansion considerably since the sequence generated from this by applying successively the operator $\xi\,\partial/\partial\xi$ does not terminate. Consequently the expansions of g_2 and f_2 no longer terminate. Thus (3.2) implies

$$g_2 = g_{20}\ln\xi + g_{22}\ln\ln\xi + g_{21} + \ldots, \tag{3.4}$$

where g_{20}, g_{21} are unchanged and

$$g_{22} = 2B_1\alpha_{12}(1-\bar{g}_2). \tag{3.5}$$

The expansion for f_2 must start in the form

$$f_2 = f_{20}\ln^2\xi + f_{23}\ln\xi\ln\ln\xi + f_{21}\ln\xi + f_{24}(\ln\ln\xi)^2 + f_{25}\ln\ln\xi + f_{22} + f_{26}\frac{\ln\ln\xi}{\ln\xi} + \frac{f_{27}}{\ln\xi} + \ldots. \tag{3.6}$$

f_{20} is still described by (2.13); f_{23} satisfies the equation

$$f_{23}''' - \tfrac{1}{2}\eta^3 f_{23}'' + 3\eta^2 f_{23}' - 5\eta f_{23} = -8\alpha_{10}\alpha_{12}\eta^2,$$

so that

$$f_{23} = \alpha_{23}\eta^2 - \tfrac{2}{15}\alpha_{10}\alpha_{12}\eta^5. \tag{3.7}$$

f_{21} satisfies (2.14) and we write its solution in the form

$$f_{21} = \alpha_{21}\eta^2 - \tfrac{2}{15}\alpha_{10}\alpha_{11}\eta^5 + \mathscr{F}_{21}(\alpha_{10}, B;\eta), \tag{3.8}$$

where $\mathscr{F}_{21}$ is the particular integral generated by

$$-2\alpha_{10}^2\eta^2 + \tfrac{1}{3}B_1\alpha_{10}\eta^4 + \tfrac{1}{5}\alpha_{10}^2\eta^6 + 2B_1\alpha_{10}(1-\bar{g}_2).$$

f_{24} satisfies a simple equation and has solution

$$f_{24} = \alpha_{24}\eta^2 - \tfrac{1}{15}\alpha_{12}^2\eta^5. \tag{3.9}$$

The next term is f_{25} which is described by

$$f_{25}''' - \tfrac{1}{2}\eta^3 f_{25}'' + 3\eta^2 f_{25}' - 5\eta f_{25} = -8\alpha_{11}\alpha_{12}\eta^2 - 2\alpha_{10}\alpha_{12}\eta^2 + 2B_1\alpha_{12}(1-\bar{g}_2) + \tfrac{1}{3}B_1\alpha_{12}\eta^4 + \tfrac{1}{5}\alpha_{10}\alpha_{12}\eta^6. \tag{3.10}$$

The phenomenon of (2.17) appears here, the choice of α_{10} given by (2.16) ensuring that (3.10) has a solution regardless of the value of α_{12}.

Turning now to the $O(1)$ balance, which earlier gave difficulties,

$$f_{22}''' - \tfrac{1}{2}\eta^3 f_{22}'' + 3\eta^2 f_{22}' - 5\eta f_{22} = -2\alpha_{10}\alpha_{12}\eta^2 - 2\alpha_{10}\alpha_{11}\eta^2 - 4\alpha_{11}^2\eta^2 + \tfrac{1}{6}B_1\alpha_{10}\eta^4 - \frac{B_1\alpha_{10}}{2\cdot 8^{\frac{1}{4}}\Gamma(\frac{1}{4})}h_2(\eta) + \eta\mathscr{F}_{21} - \tfrac{1}{2}\eta^2\mathscr{F}_{21}' + [\tfrac{1}{3}B_1\alpha_{11}\eta^4 + 2B_1\alpha_{11}(1-\bar{g}_2) + \tfrac{1}{5}\alpha_{10}\alpha_{11}\eta^6] + \tfrac{1}{5}\alpha_{10}\alpha_{12}\eta^6. \tag{3.11}$$

† We write $\ln\ln\xi$ as a shorthand for $\ln|\ln\xi|$ throughout.

The α_{11} terms do not contribute to the integral restraint associated with this equation so that α_{12} is determined in terms of B_1. Thus

$$\alpha_{12} = [1-2\ln 2]\,\alpha_{10}. \tag{3.12}$$

α_{11}, the $O(-X)^{\frac{1}{2}}$ contribution to the skin friction, is not determined by the preceding analysis or anything that follows. This is not surprising since everything we have done so far should reduce to the incompressible case when $B_1 = 0$.

The remaining difficulties with the expansion of f_2 are easily taken care of. Following the $O(1)$ balance, f_{26} satisfies

$$f_{26}''' - \tfrac{1}{2}\eta^3 f_{26}'' + 3\eta^2 f_{26}' - 5\eta f_{26} = -2\alpha_{12}^2\eta^2 + \tfrac{1}{5}\alpha_{12}^2\eta^6 \tag{3.13}$$

and this equation only has a solution when

$$\alpha_{12} = 0.$$

To avoid this conclusion another term must be added to f_1. This has to be an $O(\ln\ln\xi/\ln\xi)$ term. Similarly, the equation for f_{27}, the coefficient of the $O(1/\ln\xi)$ term in the expansion of f_2, only has a solution if an $O(1/\ln\xi)$ term is added to f_1. Now these additional terms imply that the expansion of f_2 must continue as

$$f_2 = \ldots + f_{29}\frac{(\ln\ln\xi)^2}{\ln^2\xi} + f_{210}\frac{\ln\ln\xi}{\ln^2\xi} + \ldots$$

and like terms must be added to f_1 and so on.

In order to provide reassurance that everything works out properly let us consider the precise effects of the next two terms in f_1. Equation (3.2) is replaced by

$$f_1 = f_{10}\ln\xi + f_{12}\ln\ln\xi + f_{11} + f_{13}\frac{\ln\ln\xi}{\ln\xi} + \frac{f_{14}}{\ln\xi} + O\left(\frac{\ln\ln\xi}{\ln\xi}\right)^2 \tag{3.14}$$

and to (3.4) must be added the additional terms

$$g_2 = \ldots + g_{24}\frac{\ln\ln\xi}{\ln\xi} + \frac{g_{23}}{\ln\xi} + \ldots.$$

f_{13}, like all the f_{1j} terms other than f_{11}, is simply

$$f_{13} = \alpha_{13}\eta^2, \tag{3.15}$$

so that

$$g_{24} = 2B_1\alpha_{13}(1-\bar{g}_2). \tag{3.16}$$

f_{13} effects not only the equation for f_{26} but also some of the earlier ones, but fortunately in a way that does not interfere with the integral restraints. Thus a term $-8\alpha_{10}\alpha_{13}\eta^2$ must be added to the right side of equation (3.10) for f_{25}. Also a new term

$$f_{28}\frac{(\ln\ln\xi)^2}{\ln\xi}$$

is generated, satisfying

$$f_{28}''' - \tfrac{1}{2}\eta^3 f_{28}'' + 3\eta^2 f_{28}' - 5\eta f_{28} = -8\alpha_{12}\alpha_{13}\eta^2, \tag{3.17}$$

but most important of all, the equation for f_{26} becomes

$$f_{26}''' - \tfrac{1}{2}\eta^3 f_{26}'' + 3\eta^2 f_{26}' - 5\eta f_{26} = -2\alpha_{12}^2\eta^2 - 8\alpha_{11}\alpha_{13}\eta^2 + \tfrac{1}{5}\alpha_{12}^2\eta^6 + 2B_1\alpha_{13}(1-\bar{g}_2) + \tfrac{1}{3}B_1\alpha_{13}\eta^4, \tag{3.18}$$

whence
$$\alpha_{13} = -\frac{\alpha_{12}^2}{B_1}\frac{[\Gamma(\frac{1}{4})]^3}{2\pi^{\frac{1}{2}}\Gamma(\frac{3}{4})}. \tag{3.19}$$

Continuing,
$$f_{14} = \alpha_{14}\eta^2 \tag{3.20}$$

and
$$g_{23} = 2B_1\alpha_{14}(1-\bar{g}_2) - \frac{B_1\alpha_{12}}{2\cdot 8^{\frac{1}{4}}\Gamma(\frac{1}{4})}h_2(\eta), \tag{3.21}$$

so that $-8\alpha_{10}\alpha_{14}\eta^2$ must be added to the equation for f_{22}; $-8\alpha_{12}\alpha_{14}\eta^2$ to (3.18); and f_{27} satisfies

$$f_{27}''' - \tfrac{1}{2}\eta^3 f_{27}'' + 3\eta^2 f_{27}' - 5\eta f_{27} = -2\alpha_{11}\alpha_{12}\eta^2 - 2\alpha_{10}\alpha_{13}\eta^2 - 8\alpha_{11}\alpha_{14}\eta^2 + g_{23} + \tfrac{1}{6}B_1\alpha_{12}\eta^4 + \tfrac{1}{3}B_1\alpha_{14}\eta^4 - \tfrac{1}{2}\eta^2 f_{25}' + \eta f_{25}. \tag{3.22}$$

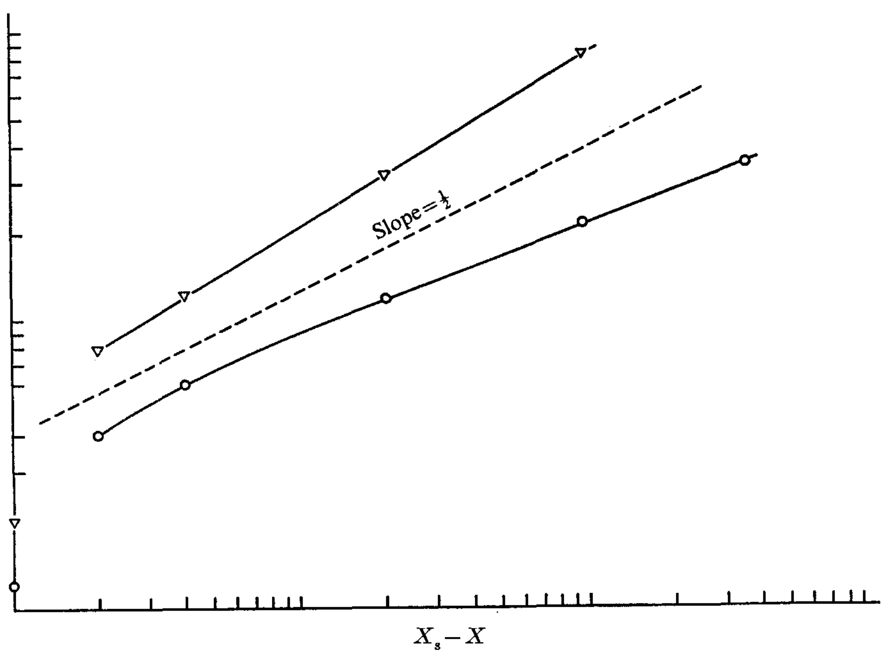

FIGURE 1. Heat transfer and skin-friction close to separation. o, heat transfer from Merkin $[10^{-6}, 10^{-3}] \times [10^{-3}, 10^{-1}]$; ▿, skin friction from Merkin $[10^{-6}, 10^{-3}] \times [10^{-4}, 10^{-2}]$.

α_{14} is now defined. It is related not only to B_1 but also to α_{11}, the other arbitrary constant at this stage. We could continue indefinitely. All of the constants α_{1j} are related to B_1, α_{11} and since none of this structure is needed for the incompressible problem they must all vanish when $B_1 = 0$. Our knowledge of the skin friction close to separation is therefore very rich with the first infinite number of terms containing only two arbitrary constants. Unfortunately, successive terms in (3.14) decrease so slowly that this knowledge is of little value. Numerical work will merely show the skin-friction behaving approximately like $(-X)^{\frac{1}{2}}$. This conclusion is in agreement with what appears to be the only published accurate numerical integration of the compressible boundary-layer equations to separation (previous computations have not been close enough, apparently).

This is the work of Merkin (1969) who considered a convection problem with flow over a vertical plate. His equations are very similar to (1.1), (1.2) and his results show singular behaviour. In figure 1 the skin friction and heat transfer are shown on a log–log plot and it is clear that both behave approximately like $(-X)^{\frac{1}{2}}$. Note that Merkin's tabulated results appear to be in error for points very close to separation.

4. Higher-order terms

The solution to second order is not completely determined by the analysis of §3 since the infinite set of constants α_{2j} are presently undefined. They are determined by an examination of the third-order solution. The terms on the right of equations (1.8) imply that the third-order solution must be of the form

$$g_3 = g_{30}\ln^2\xi + g_{31}\ln\xi\ln\ln\xi + g_{32}\ln\xi + g_{33}(\ln\ln\xi)^2 + \ldots, \tag{4.1}$$

$$f_3 = f_{30}\ln^3\xi + f_{31}\ln^2\xi\ln\ln\xi + f_{32}\ln^2\xi + f_{33}\ln\xi(\ln\ln\xi)^2 + \ldots. \tag{4.2}$$

Appropriate solutions for the g_{3j} can be found without any difficulty in principle but each of the equations for f_{3j} leads to an integral condition that determines one of the α_{2i}. It is to be expected that all the α_{2i} can be found in this way since there is no arbitrariness at this stage in the incompressible problem. The leading term for the temperature satisfies

$$g''_{30} - \tfrac{1}{2}\eta^3 g'_{30} + \tfrac{3}{2}\eta^2 g_{30} = 3B_1\alpha_{20}\eta^2 - 8B_1\alpha_{10}^2\eta^2\bar{g}'_2 - 8B_1\alpha_{10}^2\eta(1-\bar{g}_2), \tag{4.3}$$

where α_{20} is determined from the equation for f_{30},

$$f'''_{30} - \tfrac{1}{2}\eta^3 f''_{30} + \tfrac{7}{2}\eta^2 f'_{30} - 6\eta f_{30} = -10\alpha_{10}\alpha_{20}\eta^2 - \tfrac{4}{3}\alpha_{10}^3\eta^5. \tag{4.4}$$

No algebraic particular integral with a double zero can account for the η^2 term so that α_{20} is determined by the requirement that f_{30} has the appropriate behaviour, and furthermore α_{20} is non-zero. Since the equation for f_{3j} always contains a term proportional to $\alpha_{10}\alpha_{2j}\eta^2$ it seems probable that all the α_{2j} can be found in this manner. Since in general g'_{3j} does not vanish on the wall, this means that the first infinity of corrections to the heat transfer B_1 depend only on B_1 and α_{11}. Numerical computations for a cold wall should show the heat transfer approaching its limiting value like $(-X)^{\frac{1}{2}}$. The results of Merkin (1969) mentioned earlier confirm this, and are shown in figure 1.

Every f_{3j} calculated at the third order is arbitrary to the extent of a term

$$\alpha_{3j}\eta^2$$

and the α_{3j} have to be found from a study of f_4. Again, a lack of arbitrariness in the third-order incompressible solution implies that all the α_{3j} are determined. Now

$$g_4 = g_{40}\ln^3\xi + \ldots,$$

where

$$g''_{40} - \tfrac{1}{2}\eta^3 g'_{40} + 2\eta^2 g_{40} = 6B_1 f_{30} - B_1\eta f'_{30} + 5f_{10}g'_{30} + 5f_{20}g'_{20} - 3f'_{10}g_{30} - 2f'_{20}g_{20} \tag{4.5}$$

and

$$f_4 = f_{40}\ln^4\xi + \ldots,$$

where

$$f_{40}'''' - \tfrac{1}{2}\eta^3 f_{40}'' + 4\eta^2 f_{40}' - 7\eta f_{40} = 4f_{10}f_{30}' + 6f_{30}f_{10}'' - 8f_{10}'f_{30}' + 5f_{20}f_{20}'' - 4f_{20}'^2$$
$$= -12\alpha_{10}\alpha_{30}\eta^2 + \dots. \tag{4.6}$$

P_1 and S_1 both contribute to the fourth-order solution but this does not complicate matters.

The pattern is changed when we turn to g_5 and f_5 in order to find the α_{4j}. The reason for this is that an integral restraint is now associated with each of the g_{5j} as well as the f_{5j}. Stewartson (1962) has pointed out that this occurs whenever $n = 4r+1$ (r an integer) since the complementary function that is algebraic at infinity vanishes at the wall for these n. The source of the difficulty provides the resolution since we can add an arbitrary multiple of $(\eta - \frac{1}{10}\eta^5)$ to each g_{5j} and this arbitrariness can be used to satisfy the additional restraints. Thus we are naturally led to start the expansion for g_5 with

$$g_5 = g_{50}\ln^4\xi + \dots,$$

where

$$g_{50}'' - \tfrac{1}{2}\eta^3 g_{50}' + \tfrac{5}{2}\eta^2 g_{50} = 4f_{10}g_{40}' + 5f_{20}g_{30}' + 6f_{30}g_{20}' + 7B_1 f_{40}$$
$$- 4f_{10}'g_{40} - 3f_{20}'g_{30} - 2f_{30}'g_{20} - B_1\eta f_{40}', \tag{4.7}$$

but equation (4.7) does not have an appropriate solution (the α_{4j} are to be reserved for the solution of f_5 of course). However the equation for g_5 contains the term $\frac{1}{2}\eta^2\xi\,\partial g_5/\partial\xi$ so that if we write

$$g_5 = g_{51}\ln^5\xi + g_{50}\ln^4\xi + \dots,$$

then
$$g_{51}'' - \tfrac{1}{2}\eta^3 g_{51}' + \tfrac{5}{2}\eta^2 g_{51} = 0,$$
with solution
$$g_{51} = B_{51}(\eta - \tfrac{1}{10}\eta^5), \tag{4.8}$$
and this adds a term
$$-\tfrac{5}{2}B_{51}(\eta^3 - \tfrac{1}{10}\eta^7)$$

to the right of (4.7). B_{51} is then determined by the integral restraint associated with (4.7). Terms of order $\ln^4\xi\ln\ln\xi$ and $\ln^3\xi(\ln\ln\xi)^2$ also have to be deliberately introduced into g_5 but the rest of the terms needed appear in a natural way. For example, solution at the $O(\ln^3\xi)$ level is assured by the arbitrariness of g_{50}. All the constants B_{5j} are determined except the one that is introduced at the $O(1)$ level, since $\xi\,\partial/\partial\xi(1) = 0$. We will call this constant B_5 and it joins B_1 and α_{11} as an unknown. This difficulty with g_5 does not affect f_5. The reason for this is that the three extra orders added to g_5 do not add extra orders to f_5—those orders are already present. Thus the one-to-oneness between the α_{4j} and the f_{5j} is not disturbed and all the α_{4j} are determined.

The leading term in the expansion of f_5 is

$$f_{50}\ln^5\xi,$$

where

$$f_{50}'''' - \tfrac{1}{2}\eta^3 f_{50}'' + \tfrac{9}{2}\eta^2 f_{50}' - 8\eta f_{50} = B_{51}(\eta - \tfrac{1}{10}\eta^5) - 14\alpha_{10}\alpha_{40}\eta^2 + 5f_{20}f_{30}''$$
$$+ 6f_{30}f_{20}'' - 9f_{20}'f_{30}'. \tag{4.9}$$

B_{51} is of course already related to α_{40} by the restraint associated with (4.7).

Turning to f_6, more complications arise. This is because the equation

$$f_6''' - \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 = \eta^2$$

has a solution

$$f_6 = \tfrac{1}{60}(\eta^5 - \tfrac{1}{84}\eta^9),$$

so that, as with f_2, η^2 terms do not contribute to the integral restraints associated with the f_{6j}. Consequently it is no longer clear that the necessary degree of arbitrariness occurs at each stage via the α_{5j}. Indeed with

$$f_6 = f_{60} \ln^6 \xi + \ldots,$$

the equation for f_{60} is not solvable. Additional terms have to be added on to f_5 to resolve these difficulties, the first one being $O(\ln^6 \xi)$. We can expect that all the α_{5j}'s will be determined except for the $O(1)$ coefficient since this is arbitrary when $B_1 = 0$ (Stewartson 1962, p. 125). It is possible that what happens is similar to what happened with f_2 when α_{11} was undetermined, and a new infinite sequence of terms has to be added distinct from the extant mixture of logs and log-logs, but the details have not been checked. With a fourfold infinity of terms for the skin friction calculated in principle, and results obtained consistent with the numerical evidence, it does not seem very likely that an insuperable difficulty could arise in the expansion.

The hot wall case has not been discussed. The evidence is that the behaviour is singular for this problem also, but certainly the expansion generated here is not appropriate. The best hope of a resolution seems to this author to be a study of the integral equation that arose in the analysis mentioned at the beginning of § 3.

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